

6 EPEC

Name

Salvador Pineda Morente

Learning objectives

- Differentiate simultaneous and sequential games and the mathematical tools to solve them
- Explain the differences between SLSF, SLMF, MLSF and MLMF games and the mathematical tools to solve them
- Formulate equilibrium problems with equilibrium constraints (EPEC)
- Explain the different methods to solved EPECs and their features
- Solve EPECs using the diagonalization method with GAMS
- Explain the difference between open-loop and closed-loop problems

Bibliography

- Basic reading:
 - Su, C.L. Equilibrium Problems with Equilibrium Constraints: Stationarities, Algorithms, and Applications. Stanford University, Stanford, CA, USA. (2005)
 - Gabriel, S., Conejo. A., Fuller, D., Hobbs, B., and Ruiz, C. Complementarity modeling in energy markets. Springer, 2012 (chapter 7).
 - Pozo, D. “Stochastic bilevel games applications in electricity markets”, PhD dissertation, 2012.
 - Wogrin, S. Generation expansion planning in electricity markets with bilevel mathematical programming techniques, PhD dissertation, 2013.
- Further reading:
 - S. Wogrin, J. Barquin, and E. Centeno. Capacity expansion equilibria in liberalized electricity markets: An EPEC approach. IEEE Transactions on Power Systems, 28(2):1531-1539, 2013.
 - Hu, X., Ralph, D. Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices. Operations Research, 55(5), 2007.

📌 Example 6-1: Simultaneous versus sequential games with perfect information

Let us have a strategic game with two players, each of them with two available actions. The payoff matrix of this game is provided below.

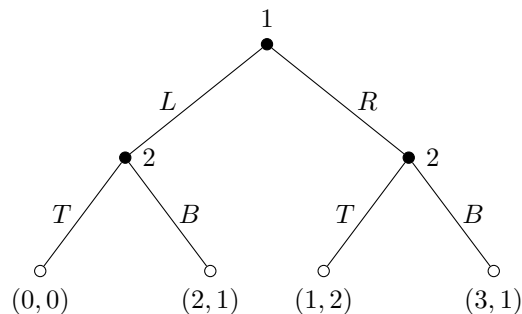
		$\mathcal{A}_1$	
		L	R
$\mathcal{A}_2$	T	(0,0)	(1,2)
	B	(2,1)	(3,1)

What is the Nash equilibrium of this game? What would be the output of this game if player 1 decides first and then player two reacts if we assume perfect information?

The Nash equilibrium is (R,T) and the payoffs are (2,1). If we have a sequential game in which player 1 decides first we can find the solution of the game as follows. Let us assume that player 1 decides L, then player 2 would decide B to maximize her payoff, and the payoff of player 1 would be 2. Let us now assume that player 1 decides R, then player 2 would decide T to maximize her payoff, and the payoff of player 1 would be only 1. Therefore, in this sequential game with perfect information, player 1 would of course decide L to get a payoff of 2. The solution of the sequential game is (L,B) and the payoffs (2,1). As you can see, this solution is different from the Nash equilibrium of the strategic game. The method we have used to find this solution is called backward induction and it is normally used to solve sequential games with perfect information.

📖 Sequential games

A sequential game is one in which players make decisions (or select a strategy) following a certain predefined order, and in which at least some players can observe the moves of players who preceded them. If no players observe the moves of previous players, then the game is simultaneous. If every player observes the moves of every other player who has gone before her, the game is one of perfect information. If some (but not all) players observe prior moves, while others move simultaneously, the game is one of imperfect information. Sequential games are represented by game trees (the extensive form) and solved using backward induction, or subgame perfect equilibrium.



📖 Subgame Perfect Nash Equilibrium

A subgame perfect Nash equilibrium is an equilibrium such that players' strategies constitute a Nash equilibrium in every subgame of the original game. It may be found by backward induction, an iterative process for solving finite extensive form or sequential games. First, one determines the optimal strategy of the player who makes the last move of the game. Then, the optimal action of the next-to-last moving player is determined taking the last player's action as given. The process continues in this way backwards in time until all players' actions have been determined.

Single-leader-single-follower game

A single-leader-single-follower (SLSF) game is stated as a bilevel optimization problem. The leader's problem is at the upper level, where the leader chooses a decision vector, x , first. After the leader has made their decision, the follower chooses their decision vector, y , solving the lower-level optimization problem (see illustration below). Single-leader-single-follower games are also known as Stackelberg games.

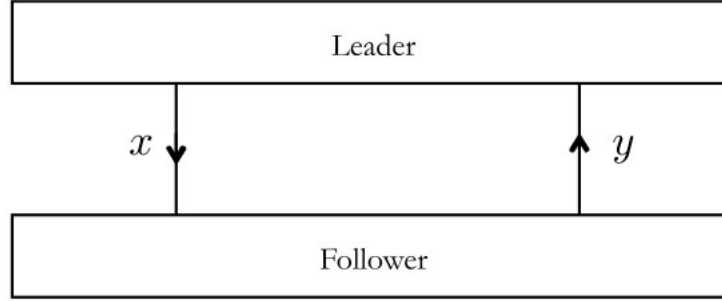


Figure taken from Pozo, D. "Stochastic bilevel games applications in electricity markets", PhD dissertation, 2012.

An example of a SLSF game is for example an investor deciding the optimal capacity of a generating unit to maximize its profit subject to an economic dispatch problems that determines dispatch quantities, flows and prices to maximize the social welfare.

As we discussed in the previous block, a Stackelberg game can be formulated as a bilevel programming problem as follows

$$\begin{aligned}
 \min_x \quad & F_0(x, y) \\
 \text{s.t.} \quad & F(x, y) \leq 0 \\
 & H(x, y) = 0 \\
 \min_y \quad & f_0(x, y) \\
 \text{s.t.} \quad & f(x, y) \leq 0 : \lambda \\
 & h(x, y) = 0 : \nu
 \end{aligned}$$

Under convexity assumption for the lower-level problem, we can replace it by its KKT conditions to formulate the following MPEC problem

$$\begin{aligned}
 \min_{x, y, \lambda, \nu} \quad & F_0(x, y) \\
 \text{s.t.} \quad & F(x, y) \leq 0 \\
 & H(x, y) = 0 \\
 & h(x, y) = 0 \\
 & \nabla_y f_0(x, y) + \lambda^T \mathbf{J}_y f(x, y) + \nu^T \mathbf{J}_y h(x, y) = 0 \\
 & 0 \leq \lambda \perp -f(x, y) \geq 0
 \end{aligned}$$

where $\mathbf{J}_y$ denote the Jacobian with respect to variables y .

Single-Leader-Multiple-Follower Games

A single-leader-multiple-follower (SLMF) game is a Stackelberg problem extension with multiple followers, where the followers are competing among themselves. In this game a single leader makes their optimal decision, x , prior to the decision of multiple followers, who are competing among themselves. Given the optimal decision of the leader, x^* , each follower makes her optimal decision, y_j^* , taking into account their competitors' optimal decisions, y_{-j}^* .

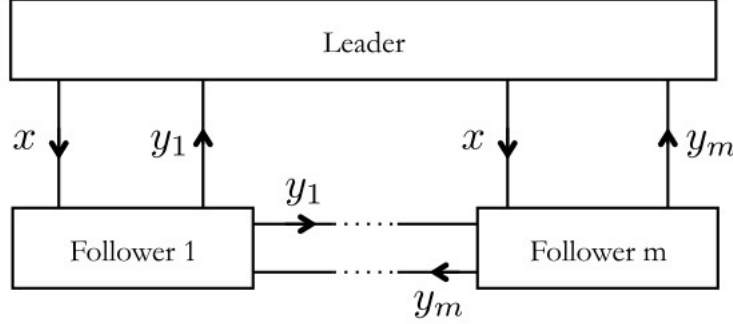


Figure taken from Pozo, D. "Stochastic bilevel games applications in electricity markets", PhD dissertation, 2012.

One example of SLMF game is a problem in which transmission investment is decided in the upper-level problem by a network planner, and each of the followers is a power producer determining its optimal generation capacity.

A SLMF game with m followers can be formulated in a generic form as follows

$$\begin{aligned}
 \min_x \quad & F_0(x, y_1, \dots, y_m) \\
 \text{s.t.} \quad & F(x, y_1, \dots, y_m) \leq 0 \\
 & H(x, y_1, \dots, y_m) = 0 \\
 & \left\{ \begin{array}{l} \min_{y_j} \quad f_0^j(x, y_j, y_{-j}) \\ \text{s.t.} \quad f^j(x, y_j, y_{-j}) \leq 0 : \lambda_j \\ \quad \quad h^j(x, y_j, y_{-j}) = 0 : \nu_j \end{array} \right\} \quad \forall j = 1, \dots, m
 \end{aligned}$$

Under the assumption that each lower-level problem is convex and satisfies the Slater condition, we can replace each of them by its corresponding KKT conditions and formulate the following MPEC problem

$$\begin{aligned}
 \min_{\substack{x, y_1, \dots, y_m \\ \lambda_1, \dots, \lambda_m, \nu_1, \dots, \nu_m}} \quad & F_0(x, y_1, \dots, y_m) \\
 \text{s.t.} \quad & F(x, y_1, \dots, y_m) \leq 0 \\
 & H(x, y_1, \dots, y_m) = 0 \\
 & h^j(x, y_j, y_{-j}) = 0, \quad \forall j = 1, \dots, m \\
 & \nabla_{y_j} f_0^j(x, y_j, y_{-j}) + \lambda_j^T \mathbf{J}_{y_j} f^j(x, y_j, y_{-j}) + \nu_j^T \mathbf{J}_{y_j} h^j(x, y_j, y_{-j}) = 0, \quad \forall j = 1, \dots, m \\
 & 0 \leq \lambda_j \perp -f^j(x, y_j, y_{-j}) \geq 0, \quad \forall j = 1, \dots, m
 \end{aligned}$$

A multiple-leader-single-follower (MLSF) game is a case when several players (leaders) anticipate simultaneously the decisions of a single player (follower). Because all the leaders make decisions at the same stage, the upper-level problem is defined as a Nash equilibrium of the leaders. Because all leader's problems depend on the competitors' decisions, the set of the n problems is coupled and complicates the resolution of this problem. EPEC techniques can be applied to solve this kind of problem.

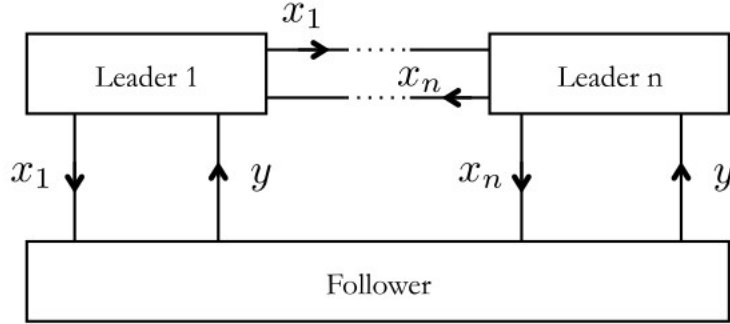


Figure taken from Pozo, D. "Stochastic bilevel games applications in electricity markets", PhD dissertation, 2012.

A MLSF game could be, for example, if we have a lot of investors determining optimal generation capacities to maximize their profits subject to an economic dispatch that determines production quantities, flows and prices to maximize the social welfare.

A MLSF with n leaders can be formulated as the following equilibrium problem

$$\left\{ \begin{array}{l} \min_{x_k} F_0^k(x_k, x_{-k}, y^k) \\ \text{s.t.} \quad F^k(x_k, x_{-k}, y^k) \leq 0 \\ H^k(x_k, x_{-k}, y^k) = 0 \\ \min_{y^k} f_0(x_1, \dots, x_n, y^k) \\ \text{s.t.} \quad f(x_1, \dots, x_n, y^k) \leq 0 : \lambda^k \\ h(x_1, \dots, x_n, y^k) = 0 : \nu^k \end{array} \right\} \quad \forall k = 1, \dots, n$$

in which lower-level variables are different for each leader k . This is so because each leader can have a different estimation of how the follower would react. Note also that if the optimal decision of the follower is a singleton for each combination of decision variables of the leader, then the lower level variables y^k would be the same for all leaders.

If the lower-level is convex and satisfies the Slater condition for any decisions of the leaders, then we can replace it by its KKT condition as follows

$$\left\{ \begin{array}{l} \min_{x_k, y^k, \lambda^k, \nu^k} F_0^k(x_k, x_{-k}, y^k) \\ \text{s.t.} \quad F^k(x_k, x_{-k}, y^k) \leq 0 \\ H^k(x_k, x_{-k}, y^k) = 0 \\ h(x_1, \dots, x_n, y^k) = 0 \\ \nabla_{y^k} f_0(x_k, x_{-k}, y^k) + \lambda^{kT} \mathbf{J}_{y^k} f(x_k, x_{-k}, y^k) + \\ \quad + \nu^{kT} \mathbf{J}_{y^k} h(x_k, x_{-k}, y^k) = 0 \\ 0 \leq \lambda^k \perp -f(x_k, x_{-k}, y^k) \geq 0 \end{array} \right\} \quad \forall k = 1, \dots, n$$

This is an EPEC problem since it is an equilibrium among players, and each of them determines the optimal strategy by solving an MPEC problem.

A multiple-leader-multiple-follower (MLMF) game is the most general instance of a bilevel game where several leaders competing among themselves have to make decisions in the first stage prior to the decisions of a set of followers competing among themselves in the second stage. The multiple-leader-multiple-follower problem is given by a set of MPEC problems, one for each leader, which is known as Equilibrium Problem with Equilibrium Constraints (EPEC).

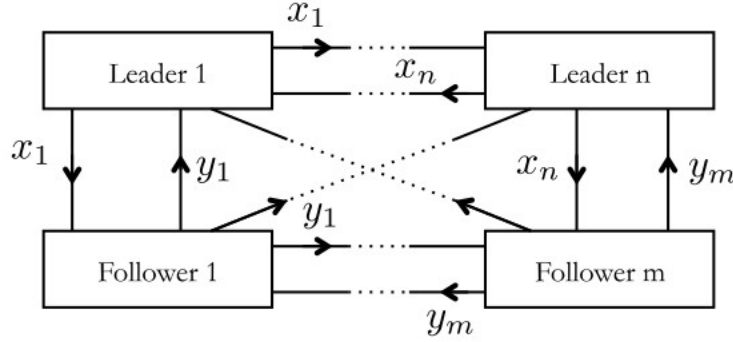


Figure taken from Pozo, D. "Stochastic bilevel games applications in electricity markets", PhD dissertation, 2012.

A MLMF game can be, for example, a generation expansion problem with Cournot power producers. The leaders would be all the investors determining the optimal capacity expansion decisions to maximize the profit. Each leader decision would affect the decision of the corresponding follower, represented as a power producer who has to determine the optimal production decisions to maximize the profit. Note that the prices and quantities are determined by solving the equilibrium of the followers.

A generic MLMF game with n leaders and m follower can be formulated as the following equilibrium problem

$$\left\{ \begin{array}{l} \min_{x_k} F_0^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) \\ \text{s.t.} \quad F^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) \leq 0 \\ H^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) = 0 \\ \left\{ \begin{array}{l} \min_{y_j^k} f_0^j(x_k, y_j^k, y_{-j}^k) \\ \text{s.t.} \quad f^j(x_k, y_j^k, y_{-j}^k) \leq 0 : \lambda_j^k \\ h^j(x_k, y_j^k, y_{-j}^k) = 0 : \nu_j^k \end{array} \right\} \forall j = 1, \dots, m \end{array} \right\} \forall k = 1, \dots, n$$

Assuming that all lower-level problems are convex and satisfy the Slater condition for any combination of leaders' decisions, they can be replaced by their KKT conditions and we obtain the following EPEC

$$\left\{ \begin{array}{l} \min_{\substack{x_k, y_1^k, \dots, y_m^k \\ \lambda_1^k, \dots, \lambda_m^k, \nu_1^k, \dots, \nu_m^k}} F_0^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) \\ \text{s.t.} \quad F^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) \leq 0 \\ H^k(x_k, x_{-k}, y_1^k, \dots, y_m^k) = 0 \\ h^j(x_k, y_j^k, y_{-j}^k) = 0, \quad \forall j = 1, \dots, m \\ \nabla_{y_j^k} f_0^j(x_k, y_j^k, y_{-j}^k) + \lambda_j^{kT} \mathbf{J}_{y_j^k} f^j(x_k, y_j^k, y_{-j}^k) + \\ \quad + \nu_j^{kT} \mathbf{J}_{y_j^k} h^j(x_k, y_j^k, y_{-j}^k) = 0, \quad \forall j = 1, \dots, m \\ 0 \leq \lambda_j^k \perp -f^j(x_k, y_j^k, y_{-j}^k) \geq 0, \quad \forall j = 1, \dots, m \end{array} \right\} \forall k = 1, \dots, n$$

Since practical applications of EPEC models often arise from MLSF or MLMF games, we consider the EPEC as the equilibrium of players that determine decisions solving an MPEC. This is illustrated in the figure below

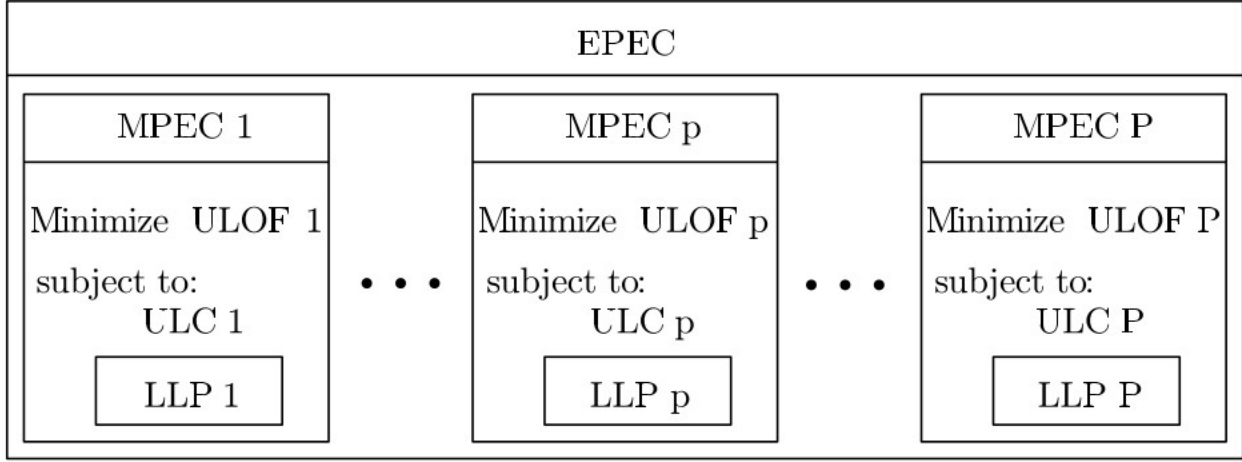


Figure taken from Ruiz, C. “Strategic offering and equilibria in electricity markets with stepwise supply functions”, PhD dissertation, 2012.

An EPEC with n leaders can be generally formulated as follows

$$\left\{ \begin{array}{l} \min_{x_k, y^k} \quad f_0^k(x_k, x_{-k}, y^k) \\ \text{s.t.} \quad \begin{array}{l} f^k(x_k, x_{-k}, y^k) \leq 0 \\ h^k(x_k, x_{-k}, y^k) = 0 \\ 0 \leq F^k(x_k, x_{-k}, y^k) \perp H^k(x_k, x_{-k}, y^k) \geq 0 \end{array} \end{array} \right\} \quad \forall k = 1, \dots, n$$

where x_k are the decisions variables of leader k , x_{-k} is the decision set of all leaders different from k , and y^k is the estimation of the followers' decisions made by leader k . We can also write each MPEC as a conventional optimization problem follows

$$\left\{ \begin{array}{l} \min_{x_k, y^k} \quad f_0^k(x_k, x_{-k}, y^k) \\ \text{s.t.} \quad \begin{array}{l} f^k(x_k, x_{-k}, y^k) \leq 0 \quad (\lambda_k) \\ h^k(x_k, x_{-k}, y^k) = 0 \quad (\nu_k) \\ -F^k(x_k, x_{-k}, y^k) \leq 0 \quad (\mu_k) \\ -H^k(x_k, x_{-k}, y^k) \leq 0 \quad (\gamma_k) \\ F^k(x_k, x_{-k}, y^k) \circ H^k(x_k, x_{-k}, y^k) = 0 \quad (\phi_k) \end{array} \end{array} \right\} \quad \forall k = 1, \dots, n$$

where $\circ$ denote the Schur product of two vector or matrices.

Methods to solve EPEC (I)

EPEC can be solved with off-the-shelf optimization software using different methods. In this course we present four of the most commonly used methods to solve EPEC problems:

1. Formulate the EPEC as a complementarity problem. This method replaces each MPEC by its KKT conditions as follows:

$$\left\{ \begin{array}{l} \nabla_{x_k} f_0^k + \lambda_k^T \mathbf{J}_{x_k} f^k + \nu_k^T \mathbf{J}_{x_k} h^k - \mu_k^T \mathbf{J}_{x_k} F^k - \gamma_k^T \mathbf{J}_{x_k} H^k + F^k \circ \phi_k^T \mathbf{J}_{x_k} H^k + H^k \circ \phi_k^T \mathbf{J}_{x_k} F^k = 0 \\ \nabla_{y^k} f_0^k + \lambda_k^T \mathbf{J}_{y^k} f^k + \nu_k^T \mathbf{J}_{y^k} h^k - \mu_k^T \mathbf{J}_{y^k} F^k - \gamma_k^T \mathbf{J}_{y^k} H^k + F^k \circ \phi_k^T \mathbf{J}_{y^k} H^k + H^k \circ \phi_k^T \mathbf{J}_{y^k} F^k = 0 \\ 0 \leq \lambda_k \perp -f^k(x_k, x_{-k}, y^k) \geq 0 \\ h^k(x_k, x_{-k}, y) = 0 \\ 0 \leq \mu_k \perp F^k(x_k, x_{-k}, y) \geq 0 \\ 0 \leq \gamma_k \perp H^k(x_k, x_{-k}, y) \geq 0 \\ F^k(x_k, x_{-k}, y) \circ H^k(x_k, x_{-k}, y) = 0 \end{array} \right\} \forall k$$

2. Sequential non-linear complementarity problems

In the previous chapter we saw how to regularize MPECs so that the KKT conditions are, at least, necessary for optimality. Following the same procedure here, we replace the equality constraint $F^k(x_k, x_{-k}, y) \circ H^k(x_k, x_{-k}, y) = 0$ by $F^k(x_k, x_{-k}, y)^T H^k(x_k, x_{-k}, y) \leq t$, with $t > 0$. The procedure is as follows:

Step 1) Set $t \leftarrow 10^9$

Step 2) Solve the following mixed non-linear complementarity problem

$$\left\{ \begin{array}{l} \nabla_{x_k} f_0^k + \lambda_k^T \mathbf{J}_{x_k} f^k + \nu_k^T \mathbf{J}_{x_k} h^k - \mu_k^T \mathbf{J}_{x_k} F^k - \gamma_k^T \mathbf{J}_{x_k} H^k + F^k \circ \phi_k^T \mathbf{J}_{x_k} H^k + H^k \circ \phi_k^T \mathbf{J}_{x_k} F^k = 0 \\ \nabla_{y^k} f_0^k + \lambda_k^T \mathbf{J}_{y^k} f^k + \nu_k^T \mathbf{J}_{y^k} h^k - \mu_k^T \mathbf{J}_{y^k} F^k - \gamma_k^T \mathbf{J}_{y^k} H^k + F^k \circ \phi_k^T \mathbf{J}_{y^k} H^k + H^k \circ \phi_k^T \mathbf{J}_{y^k} F^k = 0 \\ 0 \leq \lambda_k \perp -f^k(x_k, x_{-k}, y^k) \geq 0 \\ h^k(x_k, x_{-k}, y) = 0 \\ 0 \leq \mu_k \perp F^k(x_k, x_{-k}, y) \geq 0 \\ 0 \leq \gamma_k \perp H^k(x_k, x_{-k}, y) \geq 0 \\ 0 \leq \phi_k \perp (t - F^k(x_k, x_{-k}, y)^T H^k(x_k, x_{-k}, y)) \geq 0 \end{array} \right\} \forall k$$

Step 3) Set $t \leftarrow t/10$

Step 4) If t is lower than tolerance, stop. Otherwise, go so Step 2).

Note that with method 1 the dual variable ϕ_k is a vector, while with method 2 the dual variable ϕ_k is just a scalar.

3. Diagonalization (Jacobi). This is an iterative method to solve EPEC which is easy to implement and therefore, it is commonly used by engineers. The steps are the following:

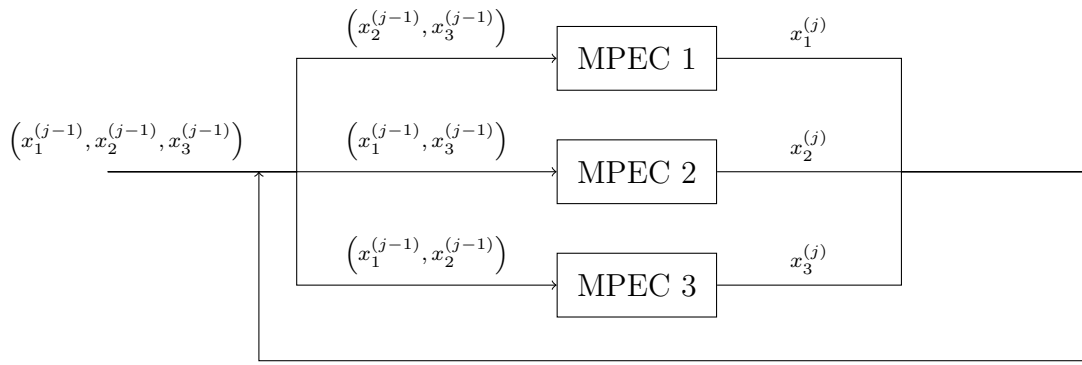
Step 1) Initialize $x_k^{(0)}$, choose a maximum number of iterations J and a tolerance ϵ . Set $j \leftarrow 0$.

Step 2) Update $j \leftarrow j + 1$. Solve one MPEC for each player k fixing the decisions of the other players $x_{-k}^{(j)}$ to the values obtained in the previous iteration $x_{-k}^{(j-1)}$ to obtain the value $x_k^{(j)}$, i.e., $x_{-k}^{(j)} = (x_1^{(j-1)}, \dots, x_{k-1}^{(j-1)}, x_{k+1}^{(j-1)}, \dots, x_n^{(j-1)})$.

Step 3) If $j > J$ stops, otherwise go to Step 4.

Step 4) Let $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)})$. If $\|\mathbf{x}^{(j)} - \mathbf{x}^{(j-1)}\| \leq \epsilon$ stops. Otherwise go to Step 2.

This method is illustrated in the figure below considering 3 leaders



4. Diagonalization (Gauss-Seidel). This is another iterative method to solve EPEC that runs as follows:

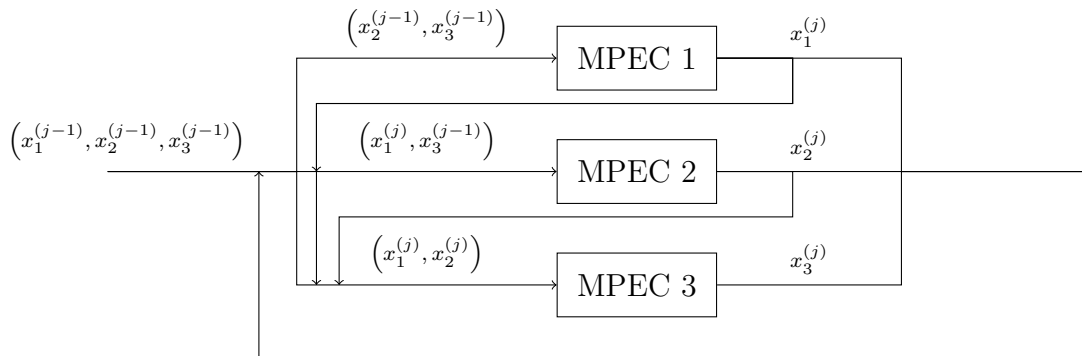
Step 1) Initialize $x_k^{(0)}$, choose a maximum number of iterations J and a tolerance ϵ . Set $j \leftarrow 0$.

Step 2) Update $j \leftarrow j + 1$. Solve one MPEC for each player k fixing the decisions of the other players as follows $x_{-k}^{(j)} = (x_1^{(j)}, \dots, x_{k-1}^{(j)}, x_{k+1}^{(j-1)}, \dots, x_n^{(j-1)})$.

Step 3) If $j > J$ stops, otherwise go to Step 4.

Step 4) Let $\mathbf{x}^{(j)} = (x_1^{(j)}, x_2^{(j)}, \dots, x_n^{(j)})$. If $\|\mathbf{x}^{(j)} - \mathbf{x}^{(j-1)}\| \leq \epsilon$ stops. Otherwise go to Step 2.

This method is illustrated in the figure below considering 3 leaders



✚ Example 6-2: Discussing the complementarity reformulation of the EPEC

Let us assume that we are solving an EPEC that has a solution, that is, there exists a set of decisions for the leaders so that none of them has an incentive to deviate. Let us assume that we solve the complementarity problem corresponding to method 1 and obtain all possible points that satisfy the KKT conditions. Are all these points equilibrium points? Can I guarantee that the solution is one of these KKT points? Justify your answer.

The answers to both questions is no. All the points that are solution to the complementarity model are not equilibrium points since the KKT conditions of the MPEC problems are not sufficient for optimality. Furthermore, since the MPEC does not satisfy the constraint qualifications, then, the KKT are not even necessary, and therefore the equilibrium solution might not be among the points that solve the complementarity problem.

✚ Example 6-3: Discussing the sequential non-linear complementarity reformulation of the EPEC

Let us assume that we are solving an EPEC that has a solution, that is, there exists a set of decisions for the leaders so that none of them has an incentive to deviate. Let us assume that we solve the sequence of complementarity problems corresponding to method 2 and obtain all possible points that satisfy the KKT conditions for $t \rightarrow 0$. Are all these points equilibrium points? Can I guarantee that the solution is one of these points? Justify your answer.

No, all the points solution to the complementarity problem for $t \rightarrow 0$ are not equilibrium because the KKT are not sufficient for optimality. Also, note that for $t > 0$, each MPEC satisfies the MFQC and therefore the KKT are necessary optimality conditions. Therefore, if the equilibrium exists, it must be among one of the solutions of the complementarity problem for $t \rightarrow 0$.

✚ Example 6-4: Discussing the diagonalization methods of the EPEC

Let us assume that we are solving an EPEC that has a solution, that is, there exists a set of decisions for the leaders so that none of them has an incentive to deviate. Let us also assume that we can adjust the values of the large constants M so that we can find the global optimal solution for each MPEC. Let us also say that we can find a set of points for which any of the two diagonalization procedures converge. Are all these points equilibrium points? Can I guarantee that the solution is one of these points? Justify your answer.

If any of the two diagonalization method converges, then we can say that we found an equilibrium solution for the EPEC since the convergence of the method only occurs when none of the players want to change their decisions. Therefore, we should be able to find the solution to the EPEC using diagonalization. The problem is that these diagonalization methods may have convergence issues and provide different solutions depending on the starting point.

✚ Example 6-5: Closed-loop generation expansion (complementarity formulation)

In Example 5-11 we formulated a bilevel problem to determine the optimal capacity $\bar{x}_g$ of one power producer anticipating the results of the market clearing. However, other power producers may also increase the capacity of their units. First, formulate the MPEC problem solved by each power producer that anticipates the market equilibrium, and then the resulting EPEC using method 1. In order to make the notation easier I recommend to use two different indexes for the power producers. Index k refers to the players deciding the investment (leaders), while index g refers to the players making production decisions (followers). Note that $k = 1, \dots, n$ and $g = 1, \dots, n$, which n being the total number of power producers. Use the same assumptions as in Example 5-11 (perfect competition, linear investment and production cost functions, linear inverse demand function, minimum production levels equal to 0).

$$\left\{ \begin{array}{ll} \min_{\bar{x}_k, x_{gt}^k, \bar{\eta}_{gt}^k, p_t^k} & i_k \bar{x}_k + \sum_t (b_k - p_t^k) x_{kt}^k \\ \text{s.t.} & -b_g + p_t^k - \bar{\eta}_{gt}^k \leq 0 \quad (\lambda_{kgt}) \quad \forall g, t \\ & -x_{gt}^k \leq 0 \quad (\mu_{kgt}) \quad \forall g, t \\ & x_{gt}^k - \bar{x}_g \leq 0 \quad (\rho_{kgt}) \quad \forall g, t \\ & -\bar{\eta}_{gt}^k \leq 0 \quad (\pi_{kgt}) \quad \forall g, t \\ & (b_g - p_t^k + \bar{\eta}_{gt}^k) x_{gt}^k = 0 \quad (\tau_{kgt}) \quad \forall g, t \\ & (\bar{x}_g - x_{gt}^k) \bar{\eta}_{gt}^k = 0 \quad (\sigma_{kgt}) \quad \forall g, t \\ & p_t^k - \alpha_t + \beta_t \sum_g x_{gt}^k = 0 \quad (\psi_{kt}) \quad \forall t \end{array} \right\} \quad \forall k$$

Note that in the third and sixth constraints, $\bar{x}_g$ is the decision variable of leader k for $g = k$, while it is a know parameter for $g \neq k$. The EPEC can then be formulated as a complementarity model as follows

$$\left\{ \begin{array}{ll} i_k - \sum_t \rho_{kgt} + \sum_t \bar{\eta}_{gt}^k \sigma_{kgt} = 0 & \forall g = k \\ b_g - p_t^k - \mu_{kgt} + \rho_{kgt} + \tau_{kgt} (b_g - p_t^k + \bar{\eta}_{gt}^k) - \sigma_{kgt} \bar{\eta}_{gt}^k + \psi_{kt} \beta_t = 0 & \forall g, t \\ -\lambda_{kgt} - \pi_{kgt} + \tau_{kgt} x_{gt}^k + \sigma_{kgt} (\bar{x}_g - x_{gt}^k) = 0 & \forall g, t \\ -x_{kt}^k + \lambda_{kgt} - \tau_{kgt} x_{gt}^k + \psi_{kt} = 0 & \forall t \\ (b_g - p_t^k + \bar{\eta}_{gt}^k) x_{gt}^k = 0 & \forall g, t \\ (\bar{x}_g - x_{gt}^k) \bar{\eta}_{gt}^k = 0 & \forall g, t \\ p_t^k - \alpha_t + \beta_t \sum_g x_{gt}^k = 0 & \forall t \\ 0 \leq \lambda_{kgt} \perp b_g - p_t^k + \bar{\eta}_{gt}^k \geq 0 & \forall g, t \\ 0 \leq \mu_{kgt} \perp x_{gt}^k \geq 0 & \forall g, t \\ 0 \leq \rho_{kgt} \perp -x_{gt}^k + \bar{x}_g \geq 0 & \forall g, t \\ 0 \leq \pi_{kgt} \perp \bar{\eta}_{gt}^k \geq 0 & \forall g, t \end{array} \right\} \quad \forall k$$

✚ Example 6-6: Closed-loop generation expansion (sequential complementarity)

Take the EPEC formulated in Example 6-5 and reformulate it using regularization for each MPEC. Explain the procedure to solve it as a sequence of complementarity problem and its advantages.

$$\left\{ \begin{array}{l} \min_{\bar{x}_k, x_{gt}^k, \bar{\eta}_{gt}^k, p_t^k} \quad i_k \bar{x}_k + \sum_t (b_k - p_t^k) x_{kt}^k \\ \text{s.t.} \quad \begin{array}{l} -b_g + p_t^k - \bar{\eta}_{gt}^k \leq 0 \quad (\lambda_{kgt}) \quad \forall g, t \\ -x_{gt}^k \leq 0 \quad (\mu_{kgt}) \quad \forall g, t \\ x_{gt}^k - \bar{x}_g \leq 0 \quad (\rho_{kgt}) \quad \forall g, t \\ -\bar{\eta}_{gt}^k \leq 0 \quad (\pi_{kgt}) \quad \forall g, t \\ \sum_{gt} (b_g - p_t^k + \bar{\eta}_{gt}^k) x_{gt}^k + \sum_{gt} (\bar{x}_g - x_{gt}^k) \bar{\eta}_{gt}^k + \\ \quad + \sum_t (p_t^k - \alpha_t + \beta_t \sum_g x_{gt}^k) \leq t \quad (\xi_k) \end{array} \end{array} \right\} \quad \forall k$$

Note that in the third and sixth constraints, $\bar{x}_g$ is the decision variable of leader k for $g = k$, while it is a known parameter for $g \neq k$. The EPEC can then be formulated as a complementarity model as follows

$$\left\{ \begin{array}{l} i_k - \sum_t \rho_{kgt} + \sum_t \bar{\eta}_{gt}^k \xi_k = 0 \quad \forall g = k \\ b_g - p_t^k - \mu_{kgt} + \rho_{kgt} + \xi_k (b_g - p_t^k + \bar{\eta}_{gt}^k - \bar{\eta}_{gt}^k + \beta_t) = 0 \quad \forall g, t \\ -\lambda_{kgt} - \pi_{kgt} + \xi_k (x_{gt}^k + \bar{x}_g - x_{gt}^k) = 0 \quad \forall g, t \\ -x_{kt}^k + \sum_g \lambda_{kgt} - \xi_k (x_{gt}^k + 1) = 0 \quad \forall t \\ 0 \leq \lambda_{kgt} \perp b_g - p_t^k + \bar{\eta}_{gt}^k \geq 0 \quad \forall g, t \\ 0 \leq \mu_{kgt} \perp x_{gt}^k \geq 0 \quad \forall g, t \\ 0 \leq \rho_{kgt} \perp -x_{gt}^k + \bar{x}_g \geq 0 \quad \forall g, t \\ 0 \leq \pi_{kgt} \perp \bar{\eta}_{gt}^k \geq 0 \quad \forall g, t \\ 0 \leq \xi_k \perp t - \sum_{gt} (b_g - p_t^k + \bar{\eta}_{gt}^k) x_{gt}^k + \sum_{gt} (\bar{x}_g - x_{gt}^k) \bar{\eta}_{gt}^k + \\ \quad + \sum_t (p_t^k - \alpha_t + \beta_t \sum_g x_{gt}^k) \geq 0 \end{array} \right\} \quad \forall k$$

The procedure to solve the EPEC is the following

- Step 1) Set $t \leftarrow 10^9$
- Step 2) Solve the complementarity model above
- Step 3) Set $t \leftarrow t/10$
- Step 4) If t is lower than tolerance, stop. Otherwise, go so Step 2).

Example 6-7: Closed-loop generation expansion (diagonalization)

Take the EPEC formulated in Example 6-5 and reformulate it using the Fortuny-Amat reformulation of each MPEC. Explain the procedure to solve it as a sequence of complementarity problem and its advantages. You can choose either the Jacobi or the Gauss-Seidel approach.

$$\left\{ \begin{array}{l} \min_{\bar{x}_k, x_{gt}^k, \bar{\eta}_{gt}^k, p_t^k, u_{gt}^k, v_{gt}^k} \quad i_k \bar{x}_k + \sum_t (b_k - p_t) x_{kt}^k \\ \text{s.t.} \quad \begin{array}{l} b_g - p_t^k + \bar{\eta}_{gt}^k \geq 0 \quad \forall g, t \\ b_g - p_t^k + \bar{\eta}_{gt}^k \leq u_{gt}^k M \quad \forall g, t \\ x_{gt}^k \geq 0 \quad \forall g, t \\ x_{gt}^k \leq (1 - u_{gt}^k) M \quad \forall g, t \\ \bar{x}_g - x_{gt}^k \geq 0 \quad \forall g, t \\ \bar{x}_g - x_{gt}^k \leq v_{gt}^k M \quad \forall g, t \\ \bar{\eta}_{gt}^k \geq 0 \quad \forall g, t \\ \bar{\eta}_{gt}^k \leq (1 - v_{gt}^k) M \quad \forall g, t \\ p_t^k - \alpha_t + \beta_t \sum_g x_{gt}^k = 0 \quad \forall t \\ u_{gt}^k, v_{gt}^k \in \{0, 1\} \quad \forall g, t \end{array} \end{array} \right\} \quad \forall k$$

According to the Jacobi diagonalization, the EPEC is solved as follows

- Step 1) Initialize $\bar{x}_k^{(0)}$, choose a maximum number of iterations J and a tolerance ϵ . Set $j \leftarrow 0$.
- Step 2) Update $j \leftarrow j + 1$. Solve the MPEC above for each player k fixing the generating capacities of the other players $\bar{x}_{-k}^{(j)}$ to the values obtained in the previous iteration $\bar{x}_{-k}^{(j-1)}$ to obtain the value $\bar{x}_k^{(j)}$, i.e., $\bar{x}_{-k}^{(j)} = (x_1^{(j-1)}, \dots, x_{k-1}^{(j-1)}, x_{k+1}^{(j-1)}, \dots, x_n^{(j-1)})$.
- Step 3) If $j > J$ stops, otherwise go to Step 4.
- Step 4) Let $\bar{\mathbf{x}}^{(j)} = (\bar{x}_1^{(j)}, \bar{x}_2^{(j)}, \dots, \bar{x}_n^{(j)})$. If $\|\mathbf{x}^{(j)} - \mathbf{x}^{(j-1)}\| \leq \epsilon$ stops. Otherwise go to Step 2.

✚ Example 6-8: Comparison of open- and closed-loop generation expansion (GenExpanClosedLoop.gms)

In Example 4-13 we formulated a generation expansion equilibrium problem assuming that capacity and production decisions are made simultaneously. We formulated that as a complementarity problem. The name for this problem is open-loop generation expansion equilibrium. After that, in Example 5-11, we proved that the sequence of decisions can actually affect the optimal decisions in generation expansion. In this example you have to compute the generation expansion equilibrium taking into account the sequence of capacity and production decisions. To do so, you have to formulate the equilibrium as an EPEC and solve it using one of the methods previously discussed. My recommendation is to implement the Jacobi diagonalization method. Use $\beta_t = 1$ and α_t varying throughout time as follows

t	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
α_t	10	20	30	40	50	60	70	80	90	100

You have to find the equilibrium for two asymmetric players as you did in Example ??-13 with the following investment and production cost functions

$$\mathcal{C}_{g_1}^I(\bar{x}_{g_1}) = 100\bar{x}_{g_1} \quad \mathcal{C}_{g_1}^P(x_{g_1t}) = 10x_{g_1t} \quad \mathcal{C}_{g_2}^I(\bar{x}_{g_2}) = 50\bar{x}_{g_2} \quad \mathcal{C}_{g_2}^P(x_{g_2t}) = 20x_{g_2t}$$

and also considering two symmetric players with the same investment and production cost functions

$$\mathcal{C}_{g_1}^I(\bar{x}_{g_1}) = 100\bar{x}_{g_1} \quad \mathcal{C}_{g_1}^P(x_{g_1t}) = 10x_{g_1t} \quad \mathcal{C}_{g_2}^I(\bar{x}_{g_2}) = 100\bar{x}_{g_2} \quad \mathcal{C}_{g_2}^P(x_{g_2t}) = 10x_{g_2t}$$

Fill in the table to compare open-loop and closed-loop results for both perfect competition and Cournot competition and comment on the obtained results.

Model	Competition	Players	$\bar{x}_{g_1}^*$	$\bar{x}_{g_2}^*$	profit $_{g_1}$	profit $_{g_2}$	SW
Open-loop	Perfect	Symmetric	25.0	25.0	0	0	7750.0
Closed-loop	Perfect	Symmetric	16.1	16.1	1557.3	1557.3	6929.6
Open-loop	Cournot	Symmetric	16.7	16.7	1722.2	1722.2	6888.9
Closed-loop	Cournot	Symmetric	16.7	16.7	1722.2	1722.2	6888.9
Open-loop	Perfect	Asymmetric	40.0	13.3	0	0	7816.7
Closed-loop	Perfect	Asymmetric	17.4	16.3	1810.0	1331.0	6828.4
Open-loop	Cournot	Asymmetric	16.7	18.3	1969.4	1477.8	6715.3
Closed-loop	Cournot	Asymmetric	19.2	17.1	2059.8	1272.4	6818.0

✚ Comparison of open- and closed-loop generation expansion (II)

- Comments for symmetric players:
 - Under perfect competition, if investment decisions are made before production decisions (closed-loop), then the producers withdraw some of the capacity to obtain a profit higher than 0. Remember that for the open-loop generation expansion (in which investment and production decisions are made simultaneously), we proved that the profits are equal to 0 for all power producers.
 - Under Cournot competition, the results of the open-loop and closed-loop models are the same. This has to do with the fact that the time periods for which the capacity constraint is binding is the same in both models. If you are interested, you can find more details about it in the following paper: Wogrin, S., Hobbs, B. F., Ralph, D., Centeno, E., Barquín, J. (2013). Open versus closed loop capacity equilibria in electricity markets under perfect and oligopolistic competition. *Mathematical Programming*, 140(2), 295–322.
 - Note that the highest social welfare is achieved for the open-loop perfect competition case for two reasons. First, because players are not exercising market power at the production stage. And second, because investment decisions are made without anticipating their impact on market outcomes.
- Comments for asymmetric players:
 - For the open-loop perfect competition model, profits are equal to 0 as we have already proven.
 - For the closed-loop perfect competition model, the players decide their investment strategically anticipating the market outcomes to obtain a profit higher than 0. It is not straight forward to know whether the player would reduce or increase their capacity a priori.
 - For the open-loop Cournot case, the players also obtain a profit higher than zero. So even though capacity and production decisions are assumed to be made simultaneously, the fact that players can exercise market power at the production stage increases their profits.
 - Again, the highest social welfare corresponds to the open-loop perfect competition case.